Fill Ups of Trigonometric Functions & Equations

Q.1. Suppose sin3x.sin 3x = $\sum_{m=0}^{n} C_m$ cos mx is an identity in x, where C_0, C_1, ...C_n are constants, and $C_n \neq 0$. then the value of n is __________ (1981 - 2 Marks)

Sol. $\sin^3 x \cdot \sin 3x = \sum_{m=0}^{n} C_m \cos mx$

$\sin^3 x.\sin 3x = \frac{1}{4}[3 \sin x - \sin 3x] \sin 3x$

$= \frac{1}{4}\left[\frac{3}{2}.2 \sin x.\sin 3x - \sin^2 3x\right]$

$= \frac{1}{4}\left[\frac{3}{2}(\cos 2x - \cos x) - \frac{1}{2}(1 - \cos 6x)\right]$

$= \frac{1}{8}[\cos 6x + 3 \cos 2x - 3 \cos x - 1]$

We observe that on LHS 6 is the max value of m.

$\therefore n = 6$

Q.2. The solution set of the system of equations $x + y = \frac{2\pi}{3}$, $\cos x + \cos y = \frac{3}{2}$ where x and y are real, is __________ . (1987 - 2 Marks)

Sol. The equations are $x + y = 2\pi/3$ ... (i)

$\cos x + \cos y = 3/2$ ...(ii)

From eq. (ii) $2\cos\frac{x+y}{2}\cos\frac{x-y}{2} = \frac{3}{2}$

$\Rightarrow 2\cos\frac{\pi}{3}\cos\frac{x-y}{2} = \frac{3}{2}$ [Using eq. (i)]

$\Rightarrow 2.\frac{1}{2}\cos\frac{x-y}{2} = \frac{3}{2} \Rightarrow \cos\frac{x-y}{2} = \frac{3}{2} > 1$

Which has no solution.

$\therefore$ The solution of given equations is φ.

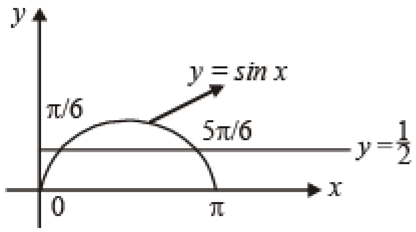
Q.3. The set of all x in the interval [0, π] for which $2\sin^2 x - 3\sin x + 1 \geq 0$, is __________ . (1987 - 2 Marks)

Sol. We have $2\sin^2 x - 3\sin x + 1 \geq 0$

$\Rightarrow (2\sin x - 1)(\sin x - 1) \geq 0$

$\Rightarrow \left(\sin x - \frac{1}{2}\right)(\sin x - 1) \geq 0 \Rightarrow \sin x \leq \frac{1}{2}$ or $\sin x \geq 1$

But we know that $\sin x \leq 1$ and $\sin x \geq 0$ for $x \in [0, \pi]$



$\Rightarrow$ either $\sin x = 1$ or $0 \leq \sin x \leq \frac{1}{2}$

$\Rightarrow$ either $x = \pi/2$ or $x \in [0, \pi/6] \cup [5\pi/6, \pi]$

Combining, we get $x \in [0, \pi/6] \cup \{\pi/2\} \cup [5\pi/6, \pi]$

Q.4. The sides of a triangle inscribed in a given circle subtend angles α, β and γ at the centre. The minimum value of the arithmetic mean of $\cos\left(\alpha + \frac{\pi}{2}\right)$, $\cos\left(\beta + \frac{\pi}{2}\right)$ and $\cos\left(\gamma + \frac{\pi}{2}\right)$ is equal to __________ (1987 - 2 Marks)

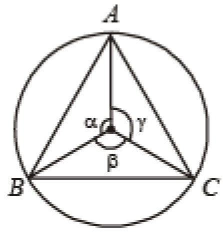
Sol. We know that A.M. $\geq$ G.M.

$\Rightarrow$ Min value of AM. is obtained when AM = GM

$\Rightarrow$ The quantities whose AM is being taken are equal.

i.e., $\cos\left(\alpha + \frac{\pi}{2}\right) = \cos\left(\beta + \frac{\pi}{2}\right)$

$= \cos\left(\gamma + \frac{\pi}{2}\right)$



$\Rightarrow \sin\alpha = \sin\beta = \sin\gamma$

Also $\alpha + \beta + \gamma = 360° \Rightarrow \alpha = \beta = \gamma = 120° = 2\pi/3$

$\therefore$ Min value of A.M.

$$= \frac{\cos\left(\frac{2\pi}{3}+\frac{\pi}{2}\right)+\cos\left(\frac{2\pi}{3}+\frac{\pi}{2}\right)+\cos\left(\cos\frac{2\pi}{3}+\frac{\pi}{2}\right)}{3}$$

$$= \frac{-\sin\frac{2\pi}{3}}{3} = -\frac{\sqrt{3}}{2}$$

Q.5.

The value of $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}\sin\frac{9\pi}{14}\sin\frac{11\pi}{14}\sin\frac{13\pi}{14}$ **is equal to** ___________

(1991 - 2 Marks)

Sol. $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}\sin\frac{9\pi}{14}\sin\frac{11\pi}{14}\sin\frac{13\pi}{14}$

$$= \left(\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\right)^2$$

$$= \left[\cos\left(\frac{\pi}{2}-\frac{\pi}{14}\right)\cos\left(\frac{\pi}{2}-\frac{3\pi}{14}\right)\cos\left(\frac{\pi}{2}-\frac{5\pi}{14}\right)\right]^2$$

$$= \left[\cos\frac{3\pi}{7}\cos\frac{2\pi}{7}\cos\frac{\pi}{7}\right]^2 = \left[\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\right]^2$$

$$= \left(\frac{1}{8\sin\pi/7}\sin\frac{8\pi}{7}\right)^2 = \left(\frac{\sin(\pi+\pi/7)}{8\sin\pi/7}\right)^2$$

$$= \left(\frac{-\sin\pi/7}{8\sin\pi/7}\right)^2 = \left(\frac{1}{8}\right)^2 = \frac{1}{64}$$

Q.6. If K = sin(π/ 18) sin(5π/ 18) sin(7π/ 18), then the numerical value of K is __________ . (1993 - 2 Marks)

Sol. $K = \sin\frac{\pi}{18}\sin\frac{5\pi}{18}\sin\frac{7\pi}{18}$

$= \cos\left(\frac{\pi}{2}-\frac{\pi}{18}\right)\cos\left(\frac{\pi}{2}-\frac{5\pi}{18}\right)\cos\left(\frac{\pi}{2}-\frac{7\pi}{18}\right)$

$= \cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}$

$= \frac{1}{2^3\sin\frac{\pi}{9}}.\sin\frac{8\pi}{9}$

[Using $\cos\alpha\cos 2\alpha\ \cos 2^2\alpha \ldots\ldots\ldots \cos 2^{n-1}\alpha$

$= \frac{1}{2^n\sin\alpha}.\sin(2^n\alpha)]$

$= \frac{1}{8\sin\pi/9}.\sin\pi/9 = \frac{1}{8}$

Q.7. If A > 0,B>0 and A +B =π/ 3, then the maximum value of tan A tan B is__________ . (1993 - 2 Marks)

Sol. $A + B = \pi/3 \Rightarrow \tan(A+B) = \sqrt{3}$

$\Rightarrow \frac{\tan A+\tan B}{1-\tan A\tan B} = \sqrt{3} \Rightarrow \frac{\tan A+\frac{y}{\tan A}}{1-y} = \sqrt{3}$

[where y = tan A tan B]

$\Rightarrow \tan^2 A + \sqrt{3}(y-1)\tan A + y = 0$

For real value of tan A, $3(y-1)^2 - 4y \geq 0$

$\Rightarrow 3y^2-10y+3\geq 0 \ \Rightarrow (y-3)(y-\frac{1}{3})\geq 0$

$\Rightarrow y\leq\frac{1}{3}$ or ≥ 3

But A, B > 0 and A + B = π/3 ⇒ A, B < π/3

$\Rightarrow$ tan A tan B < 3

$\therefore y \le \frac{1}{3}$ i.e., max. value of y is 1/3.

Q.8. General value of θ satisfying the equation $\tan^2\theta + \sec 2\theta = 1$ is_________ . (1996 - 1 Mark)

Sol. $\tan^2\theta + \sec 2\theta = 1$

$t^2 + \frac{1+t^2}{1-t^2} = 1$ where t = tanθ

$\therefore t^2(t^2 - 3) = 0 \quad \therefore \tan\theta = 0, \pm\sqrt{3}$ etc.

which means $\theta = n\pi$ and $\theta = n\pi \pm \pi/3$

Q.9. The real roots of the equation $\cos^7 x + \sin^4 x = 1$ in the interval (–π, π) are ..., ..., and _________ . (1997 - 2 Marks)

Sol. $\cos^7 x = 1 - \sin^4 x = (1 - \sin^2 x)(1 + \sin^2 x) = \cos^2 x (1 + \sin^2 x)$

$\therefore \cos x = 0$ or $x = \pi/2, -\pi/2$

or $\cos^5 x = 1 + \sin^2 x$ or $\cos^5 x - \sin^2 x = 1$

Now maximum value of each cos x or sin x is 1.

Hence the above equation will hold when cos x = 1 and sin x = 0. Both these imply x = 0

Hence $x = -\frac{\pi}{2}, \frac{\pi}{2}, 0$

True False of Trigonometric Functions & Equations

Q.1. If tan A = (1- cos B)/sinB , then tan 2A = tan B. (1983 - 1 Mark)

Ans. T

Sol. $\tan A = \frac{1-\cos B}{\sin B} = \frac{2\sin^2 B/2}{2\sin B/2 \cos/2} = \tan B/2$

Hence $\tan 2A = \frac{2\tan A}{1-\tan^2 A} = \frac{2\tan B/2}{1-\tan^2 B/2} = \tan B$

∴ Statement is true.

Q.2. There exists a value of θ between 0 and 2π that satisfies the equation $\sin^4\theta$ - 2 $\sin^2\theta$ - 1 = 0 . (1984 - 1 Mark)

Ans. F

Sol. Given equation is $\sin^4\theta - 2\sin^2\theta - 1 = 0$ Here,

D = 4 + 4 = 8

$\therefore \sin^2\theta = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}.$

But $\sin^2\theta$ can not be –ve $\therefore \sin^2\theta = \sqrt{2} + 1$

But as $-1 \le \sin\theta \le 1 \quad \therefore \sin^2\theta \ne \sqrt{2} + 1$

Thus there is no value of θ which satisfy the given equation.

∴ Statement is false.

Subjective questions of Trigonometric Functions & Equations

Q. 1. If $\tan\alpha = \frac{m}{m+1}$ **and** $\tan\beta = \frac{1}{2m+1}$**, find the possible values of (α + β).** **(1978)**

Ans. $n\pi + \frac{\pi}{4}$

Sol. We know tan $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}$

$$\Rightarrow \tan(\alpha+\beta) = \frac{\frac{m}{m+1} + \frac{1}{2m+1}}{1 - \frac{m}{m+1}\cdot\frac{1}{2m+1}} = \frac{2m^2+2m+1}{2m^2+2m+1} = 1$$

$$\Rightarrow \alpha + \beta = n\pi + \pi/4 \text{ where } n \in Z.$$

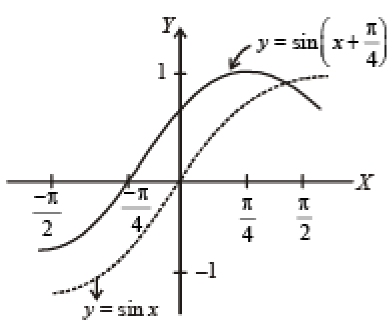
2. (a) Draw the graph of y = $\frac{1}{\sqrt{2}}$**(sinx + cosx) from x =** $-\frac{\pi}{2}$ **to** $x = \frac{\pi}{2}$.

(b) If cos $(\alpha+\beta) = \frac{4}{5}$, $\sin(\alpha-\beta) = \frac{5}{13}$ **and α, β lies between 0 and** $\frac{\pi}{4}$**, find tan2α. (1979)**

Sol. (a) Given: $y = \frac{1}{\sqrt{2}}$(sinx + cosx) sin $= \sin\left(x + \frac{\pi}{4}\right)$ ...(1)

Now, to draw the graph of $y = \sin\left(x+\frac{\pi}{4}\right)$, we first draw

the graph of y = sin x and then on shifting it by $-\frac{\pi}{4}$ we will obtain the required graph as shown in figure given below.



(b) $\cos(\alpha+\beta)=\frac{4}{5}$

$\Rightarrow \tan(\alpha+\beta)=\frac{3}{4}, 0<\alpha, \beta<\frac{\pi}{4}$

$\sin(\alpha-\beta)=\frac{5}{13} \Rightarrow \tan(\alpha-\beta)=\frac{5}{12}$

$\therefore \tan 2\alpha = \tan[(\alpha+\beta)+(\alpha-\beta)]$

$=\frac{\tan(\alpha+\beta)+\tan(\alpha-\beta)}{1-\tan(\alpha+\beta)\tan(\alpha-\beta)}$

$=\frac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}\times\frac{5}{12}}=\frac{9+5}{12}\times\frac{16}{11}=\frac{56}{33}$

3. Given $\alpha + \beta - \gamma = \pi$, prove that $\sin^2\alpha+\sin^2\beta-\sin^2\gamma=2\sin\alpha\sin\beta\cos\gamma$ **(1980)** \

Sol. Given $\alpha + \beta - \gamma = \pi$ and to prove that

$\sin^2\alpha + \sin^2\beta - \sin^2\gamma = 2\sin\alpha \sin\beta \cos\gamma$

L.H.S. $= \sin^2\alpha + \sin^2\beta - \sin^2\gamma$

[Using $\sin^2\alpha - \sin^2\beta = \sin(A+B)\sin(A-B)$]

$= \sin^2\alpha + \sin(\beta+\gamma)\sin(\beta-\gamma)$

$= \sin^2\alpha + \sin(\beta+\gamma)\sin(p-\alpha)$ $\quad (\because \alpha+\beta-\gamma=\pi)$

$= \sin^2\alpha + \sin(\beta+\gamma)\sin\alpha$

$= \sin\alpha(\sin\alpha + \sin(\beta+\gamma))$

$= \sin\alpha[\sin[\pi-(\beta-\gamma)] + \sin(\beta+\gamma)]$

$= \sin\alpha[\sin(\beta-\gamma) + \sin(\beta+\gamma)]$

$= \sin\alpha[2\sin\beta\cos\gamma] = 2\sin\alpha\sin\beta\cos\gamma =$ R.H.S.

4. Given $A=\left\{x:\frac{\pi}{6}\le x\le\frac{\pi}{3}\right\}$ **and f (x) = cos x – x (1 + x); find f (A). (1980)**

Sol.

$A=\left\{x:\frac{\pi}{6}\le x\le\frac{\pi}{3}\right\}$

f (x) = cos x – x (1+ x)

f '(x) = – sin x – 1– 2x < 0, ∀ x∈A

∴ f is a decreasing function.

$\therefore$ as $\frac{\pi}{6}\le x\le\frac{\pi}{3}\Rightarrow\ f\left(\frac{\pi}{3}\right)\le f(x)\le\left(\frac{\pi}{6}\right)$

$\Rightarrow\ \cos\frac{\pi}{3}-\frac{\pi}{3}\left(1+\frac{\pi}{3}\right)\le f(x)\le\cos\frac{\pi}{6}-\frac{\pi}{6}\left(1+\frac{\pi}{6}\right)$

$\therefore\ f(A)=\left[\frac{1}{2}-\frac{\pi}{3}\left(1+\frac{\pi}{3}\right),\ \frac{\sqrt{3}}{2}-\frac{\pi}{6}\left(1+\frac{\pi}{6}\right)\right]$

5. For all θ in [0, π/ 2] show that, cos (sinθ) ≥ sin (cos q) . (1981 - 4 Marks)

Ans. Sol. We have

$\cos\theta+\sin\theta=\sqrt{2}\left[\frac{1}{\sqrt{2}}\cos\theta+\frac{1}{\sqrt{2}}\sin\theta\right]$

$=\sqrt{2}\sin(\pi/4+\theta)$

$\therefore\ \cos\theta+\sin\theta\le\sqrt{2}<\pi/2\ \left(\because\begin{matrix}\sqrt{2}=1.414\\ \pi/2=1.57\end{matrix}\right)$

∴ cosθ + sinθ <π/ 2 ⇒ cosθ < π/ 2 - sinθ ...(1)

As q ∈ [0,π / 2] in which sinθ increases.

∴ Taking sin on both sides of eq. (1), we get

sin (cosθ) < sin (π /2 –sinθ)

sin (cosθ) < cos (sinθ)

$\Rightarrow \cos(\sin\theta) > \sin(\cos\theta)$ ....(1)

Hence the result.

6. Without using tables, prove that

(sin 12°) (sin 48°) (sin 54°) = 1/8 (1982 - 2 Marks)

Ans. Sol. L.H.S. = sin 12° sin 48° sin 54° = 1/2 [2 sin 12° cos 42°] sin 54°

$$= \frac{1}{2}\sin^2 54° - \frac{1}{2}\sin 54° = \frac{1}{4}[2\sin^2 54° - \sin 54°]$$

Now we know that $\sin 54° = \frac{1+\sqrt{5}}{4}$

$\therefore$ We get, $= \frac{1}{4}\left[2\left(\frac{1+\sqrt{5}}{4}\right)^2 - \left(\frac{1+\sqrt{5}}{4}\right)\right]$

$$= \frac{1}{4}\left[2\left(\frac{1+5+2\sqrt{5}}{16}\right) - \left(\frac{1+\sqrt{5}}{4}\right)\right]$$

$$= \frac{1}{4}\times\frac{1}{8}\left[6+2\sqrt{5}-2-2\sqrt{5}\right]$$

$$= \frac{1}{32}\times 4 = \frac{1}{8} = \text{R.H.S.}$$

7. Show that 16cos $\left(\frac{2\pi}{15}\right)\cos\left(\frac{4\pi}{15}\right)\cos\left(\frac{8\pi}{15}\right)\cos\left(\frac{16\pi}{15}\right) = 1$ (1983 - 2 Marks)

Ans. Sol. We know that,

cos A cos 2A cos 4A . . . . cos 2^n

$$A = \frac{1}{2^{n+1}\sin A}\sin(2^{n+1}A)$$

$$\therefore 16\cos\frac{2\pi}{15}\cos 2\left(\frac{2\pi}{15}\right)\cos 2^2\left(\frac{2\pi}{15}\right)\cos 2^3\left(\frac{2\pi}{15}\right)$$

$= 16 . \frac{\sin(2^4 A)}{2^4 \sin A}$ (where A = 2π/15)

$$= 16 . \frac{\sin(32\pi/15)}{16\sin 2\pi/15} = \frac{\sin(32\pi/15)}{\sin(2\pi + 2\pi/15)} = \frac{\sin(32\pi/15)}{\sin(32\pi/15)} = 1$$

8. Find all the solution of 4 cos ^{2} x sin x - 2 sin^2 x= 3 sinx **(1983 - 2 Marks)**

Ans. Sol. Given eq. is,

$4\cos^2 x \sin x - 2\sin^2 x = 3\sin x$

$\Rightarrow 4\cos^2 x \sin x - 2\sin^2 x - 3\sin x = 0$

$\Rightarrow 4(1 - \sin^2 x)\sin x - 2\sin^2 x - 3\sin x = 0$

$\Rightarrow \sin x\,[4\sin^2 x + 2\sin x - 1] = 0$

$\Rightarrow$ either $\sin x = 0$ or $4\sin^2 x + 2\sin x - 1 = 0$

If $\sin x = 0 \Rightarrow x = np$

$\Rightarrow$ If $4\sin^2 x + 2\sin x - 1 = 0 \Rightarrow \sin x = = \frac{-1\pm\sqrt{5}}{4}$

If $\sin x = \frac{-1\pm\sqrt{5}}{4} = \sin 18^\circ = \sin\frac{\pi}{10}$

then $x = nx + (-1)^4\frac{\pi}{10}$

If $\sin x = -\left(\frac{\sqrt{5}+1}{4}\right) = \sin(-54^O) = \sin\left(\frac{-3\pi}{10}\right)$

then $x = n\pi + (-1)^n\left(\frac{-3\pi}{10}\right)$

Hence, $x = n\pi,\ n\pi + (-1)^n\frac{\pi}{10}$ or $n\pi + (-1)^n\left(\frac{-3\pi}{10}\right)$

where n is some integer

9. Find the values of $x \in (-\pi, +\pi)$ which satisfy the equation

$8^{(1+|\cos x|+|\cos^2 x|+|\cos^3 x|+......)} = 4^3$ **(1984 - 2 Marks)**

Sol. The given equation is

$$8^{(1+|\cos x|+|\cos^2 x|+|\cos^3 x|+....)} = 4^3$$

$$\Rightarrow 2^{3(1+|\cos x|+|\cos^2 x|+|\cos^3 x|+....)} = 2^6$$

$$\Rightarrow 3\,(1+|\cos x|+|\cos^2 x|+|\cos^3 x|+...) = 6$$

$$\Rightarrow 1+|\cos x|+|\cos^2 x|+|\cos^3 x|+.... = 2$$

$$\Rightarrow \frac{1}{1-|\cos\ x|} = 2$$ **NOTE THIS STEP**

$$\Rightarrow 1-\cos x = 1/2 \Rightarrow |\cos x| = \frac{1}{2}$$

$$\Rightarrow x = \pi/3, -\pi/3, 2\pi/3, -2\pi/3, ...$$

The values of $x \in (-\pi, \pi)$ are $\pm\ \pi/3,\ \pm\ 2\pi/3$.

10. Prove that tanα + 2 tan 2α + 4 tan 4α + 8 cot 8α = cotα (1988 - 2 Marks)

Ans. Sol. We know that tan $2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$

$$\Rightarrow \frac{1-\tan^2\alpha}{\tan\alpha} = 2\cot 2\alpha \Rightarrow \cot\alpha - \tan\alpha = 2\cot 2\alpha$$

Now we have to prove

tanα + 2 tan 2α + 4 tan 4α + 8 cot 8α = cot α

LHS

tan α + 2 tan 2α + 4 tan 4α + 4 (2 cot 2 . 4α)

= tan α + 2 tan 2α + 4 tan 4α + 4 (cot 4α - tan 4α) [Using (1)]

= tan α + 2 tan 2α + 4 tan 4α + 4 cot 4α - 4 tan 4α

= tan α + 2 tan 2α + 2 (2 cot 2. 2 a)

= tan α + 2 tan 2α + 2 (cot α – tan 2α)

$= \tan\alpha + 2\tan 2\alpha + 2(2\cot 2\alpha - \tan 2\alpha)$ [Using (1)]

$= \tan\alpha + 2\cot 2\alpha$

$= \tan\alpha + (\cot\alpha - \tan\alpha)$ [Using (1)]

$= \cot\alpha$ = RHS.

11. ABC is a triangle such that $\sin(2A + B) = \sin(C – A) = – \sin(B + 2C) = 1/2$
If A, B and C are in arithmetic progression, determine the values of A, B and C. (1990 - 5 Marks)

Ans. Sol. Given that in ΔABC, A, B and C are in A.P.

$\therefore A + C = 2B$

also $A + B + C = 180° \Rightarrow B + 2B = 180° \Rightarrow B = 60°$

Also given that, $\sin(2A + B) = \sin(C – A) = – \sin(B + 2C) =$

$\Rightarrow \sin(2A + 60°) = \sin(C – A) = – \sin(60 + 2C) = \mathbf{1/2}$ ..(1)

From eq. (1), we have

$\sin(2A + 60°) = \mathbf{1/2}$ $\Rightarrow 2A + 60° = 30°, 150°$

but A can not be –ve

$\therefore 2A + 60° = 150° \Rightarrow 2A = 90° \Rightarrow A = 45°$

Again from (1) $\sin(60° + 2C) = \mathbf{- 1/2}$

$\Rightarrow 60° + 2C = 210°$ or $330°$

$\Rightarrow C = 75°$ or $135°$

Also from (1) $\sin(C – A) = \mathbf{1/2} \Rightarrow C – A = 30°, 150°$

For $A = 45°, C = 75°$ or $195°$ (not possible) $\therefore C = 75°$

Hence we have $A = 45°, B = 60°, C = 75°$

12. If exp {($\sin^2 x + \sin^4 x + \sin^6 x +$ .... ∞) In 2} satisfies the equation $x^2 - 9x + 8 = 0$, find the value of $\frac{\cos x}{\cos x + \sin x}, 0 < x < \frac{\pi}{2}$. (1991 - 4 Marks)

Sol. Let $y = \exp[\sin^2 x + \sin^4 x + \sin^6 x + \ldots \infty] \ln 2$

$$= e^{\ln \cdot 2^{\sin^2 x + \sin^4 x + \sin^6 x + \ldots \infty}}$$

$$= 2^{\sin^2 x + \sin^4 x + \sin^6 x + \ldots \infty} = 2^{\frac{\sin^2 x}{1 - \sin^2 x}} = 2^{\tan^2 x}$$

As y satisfies the eq.

$x^2 - 9x + 8 = 0 \qquad \therefore y^2 - 9y + 8 = 0$

$\Rightarrow (y - 1)\ (y - 8) = 0 \Rightarrow y = 1, 8$

$\Rightarrow\ 2^{\tan^2 x} = 1$ or $2^{\tan^2 x} = 8$

$\Rightarrow \tan 2\, x = 0$ or $\tan 2x = 3$

$\Rightarrow \tan x = 0$ or $\tan x = \sqrt{3}, -\sqrt{3}$

$\Rightarrow x = 0$ or $x = \pi/3,\ 2\pi/3$

But given that $0 < x < \pi/2 \Rightarrow x = \pi/3$

Hence $\frac{\cos x}{\cos x + \sin x} = \frac{1}{1 + \tan x} = \frac{1}{1 + \sqrt{3}} = \frac{\sqrt{3} - 1}{2}$

13. Show that the value of $\frac{\tan x}{\tan 3x}$ wherever defined never lies between 1/3 and 3. (1992 - 4 Marks)

Sol.

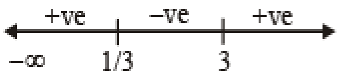
Let $y = \frac{\tan x}{\tan 3x} \Rightarrow \quad y = \frac{\tan x(1 - 3\tan^2 x)}{3\tan x - \tan^3 x}$

$\Rightarrow 3y - 3\tan^2 x = 1 - 3\tan^2 x$

$\Rightarrow (y - 3)\tan^2 x = 3y - 1 \Rightarrow \tan^2 x = \frac{3y-1}{y-3}$

$\Rightarrow \frac{3y-1}{y-3} > 0$ (L.H.S. being a prefect square)

$\Rightarrow \frac{(3y-1)(y-3)}{(y-3)^2} > 0 \Rightarrow (3y - 1)(y - 3) > 0$



$\Rightarrow y < \frac{1}{3}$ or $y > 3$

Thus y never lies between $\frac{1}{3}$ and 3

14. Determine the smallest positive value of x (in degrees) for which tan(x + 100°) = tan (x + 50°) tan(x) tan (x – 50°). (1993 - 5 Marks)

Ans. Sol. Given that, tan (x + 100°) = tan (x + 50°) tan x tan (x – 50°)

$\Rightarrow \frac{\tan(x+100^\circ)}{\tan x} = \tan(x + 50^\circ)\tan(x - 50^\circ)$

$\Rightarrow \frac{\sin(x+100^\circ)\cos x}{\cos(x+100^\circ)\sin x} = \frac{\sin(x+50^\circ)\sin(x-50^\circ)}{\cos(x+50^\circ)\cos(x-50^\circ)}$

$\Rightarrow \frac{\sin(2x+100^\circ)+\sin 100^\circ}{\sin(2x+100^\circ)-\sin 100^\circ} = \frac{\cos 100^\circ - \cos 2x}{\cos 100^\circ + \cos 2x}$

Applying componendo and dividendo, we get

$\Rightarrow \frac{2\sin(2x+100^\circ)}{2\sin 100^\circ} = \frac{2\cos 100^\circ}{-2\cos 2x}$

$\Rightarrow 2\sin(2x + 100^\circ)\cos 2x = -2\sin 100^\circ \cos 100^\circ$

$\Rightarrow \sin(4x + 100^\circ) + \sin 100^\circ = -\sin 200^\circ$

$\Rightarrow \sin(4x + 10^\circ + 90^\circ) + \sin(90^\circ + 10^\circ) = -\sin(180 + 20^\circ)$

$\Rightarrow \cos(4x + 10^\circ) + \cos 10^\circ = \sin 20^\circ$

$\Rightarrow \cos(4x + 10^\circ) = \sin 20^\circ - \cos 10^\circ$

$\Rightarrow \cos(4x + 10^\circ) = \sin 20^\circ - \sin 80^\circ$

$= -2\cos 50° \sin 30° = -2\cos 50° . \frac{1}{2} = -\cos 50° = \cos 130°$

$\Rightarrow 4x + 10° = 130° \Rightarrow x = 30°$

15. Find the smallest positive number p for which the equation cos(p sin x) = sin(p cos x) has a solution $x \in [0,2\pi]$. (1995 - 5 Marks)

Sol. Given that $\cos\theta = \sin\varphi$

where $\theta = p \sin x$, $\varphi = p \cos x$

Above is possible when both $\theta = \phi = \frac{\pi}{4}$ or $\theta = \phi = \frac{5\pi}{4}$

$\therefore p \sin x = \frac{\pi}{4}$ or $p \sin x = \frac{5\pi}{4}$

and $p \cos x = \frac{\pi}{4}$ or $p \cos x = \frac{5\pi}{4}$

Squaring and adding, $p^2 = \frac{\pi^2}{16} . 2 \text{or} \frac{25\pi^2}{16} . 2$

$\therefore \quad p = \frac{\pi}{4}\sqrt{2}$ only for least positive value or $p = \frac{\pi}{4}\sqrt{2}$

16. Find all values of θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ satisfying the equation $(1 - \tan\theta)(1 + \tan\theta) \sec^2\theta + 2^{\tan^2\theta} = 0.$ (1996 - 2 Marks)

Sol. Given :

$(1 - \tan\theta)(1 + \tan\theta) \sec^2\theta + 2^{\tan^2\theta} = 0$

or $(1 - \tan^2\theta)(1 + \tan^2\theta) + 2^{\tan^2\theta} = 0$

Let us put $\tan^2\theta = t$

$\therefore (1 - t)(1 + t) + 2^t = 0$ or $1 - t^2 + 2^t = 0$

It is clearly satisfied by t = 3.
as – 8 + 8 = 0 $\therefore$ $\tan^2\theta = 3$

$\therefore$ p = $\pm$ $\pi/3$ in the given interval.

17. Prove that the values of the function $\frac{\sin x \cos 3x}{\sin 3x \cos x}$, do not lie between 1/3 and 3 for any real x. (1997 - 5 Marks)

Sol. Let $y = \frac{\sin x \cos 3x}{\sin 3x \cos x} = \frac{\tan x}{\tan 3x}$

We have $y = \frac{\tan x}{\tan 3x} = \frac{\tan x(1-3\tan^2 x)}{3\tan x - \tan^3 x} = \frac{1-3\tan^2 x}{3-\tan^2 x}$

(the expression is not defined if tan x = 0)

$\Rightarrow 3y - (\tan^2 x)\, y = 1 - 3\tan^2 x \Rightarrow 3y - 1 = (y-3)\tan^2 x$

$\Rightarrow \quad \tan^2 x = \frac{3y-1}{y-3} = \frac{(3y-1)(y-3)}{(y-3)^2}$

Since $\tan^2 x > 0$, we get $(3y-1)(y-3) > 0$

$\Rightarrow \quad \left(y - \frac{1}{3}\right)(y-3) > 0 \Rightarrow y < \frac{1}{3}$ or $y > 3$

This shows that y cannot lie between $\frac{1}{3}$ and 3.

18. Prove that $\sum_{k=1}^{n-1} (n-k) \cos \frac{2k\pi}{n} = -\frac{n}{2}$ where n $\geq$ 3 is an integer. (1997 - 5 Marks)

Sol. Expanding the sigma on putting k = 1, 2, 3, ..., n

$S = (n-1) \cos \frac{2\pi}{n} + (n-2) \cos 2 . \frac{2\pi}{n} + ...$

$+ 1.\cos (n-1) \cos \frac{2\pi}{n}$ ... (1)

We know that $\cos\theta = \cos(2\pi - \theta)$

Replacing each angle θ by $2\pi - \theta$ in (1), we get

$S = (n-1)\cos(n-1)\frac{2\pi}{n} + (n-2)\cos(n-2)\frac{2\pi}{n} + ...+ 1.\cos\frac{2\pi}{n}$ by (1) .....(2)

Add terms in (1) and (2) having the same angle and take n common

$\therefore 2S = n\left[\cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \cos\frac{6\pi}{n} + ...+ \cos(n-1)\frac{2\pi}{n}\right]$

Angles are in A.P. of $d = \frac{2\pi}{n}$

$$2S = n\left[\frac{\sin(n-1)\frac{\pi}{n}}{\sin\frac{\pi}{n}}\cos\frac{\frac{2\pi}{n}+(n-1)\frac{2\pi}{n}}{2}\right]$$ **NOTE THIS STEP**

$= n .1 \cos\pi = -n \because \sin(\pi - \theta) = \sin\theta \therefore S = -n/2$

19. In any triangle ABC, prove that (2000 - 3 Marks)

$\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$.

Ans. Sol. We have, $A + B + C = \pi$

$\Rightarrow \frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2} \Rightarrow \frac{A}{2} + \frac{B}{2} = \frac{\pi}{2} - \frac{C}{2}$

or $\cot\left(\frac{A}{2} + \frac{B}{2}\right) = \cot\left(\frac{\pi}{2} - \frac{C}{2}\right)$

$$\Rightarrow \frac{\cot\frac{A}{2}.\cot\frac{B}{2} - 1}{\cot\frac{A}{2} + \cot\frac{B}{2}} = \tan\frac{C}{2}$$

$$\Rightarrow \cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}.\cot\frac{B}{2}.\cot\frac{C}{2}$$

20. Find the range of values of t for which $2\sin t = \frac{1-2x+5x^2}{3x^2-2x-1}$, $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(2005 - 2 Marks)

Ans. Sol. Given that, $2\sin t = \frac{1-2x+5x^2}{3x^2-2x-1}$, $t \in [-\pi/2, \pi/2]$

This can be written as

$(6\sin t - 5)x^2 + 2(1 - 2\sin t)x - (1 + 2\sin t) = 0$

For given equation to hold, x should be some real number, therefore above equation should have real roots i.e., $D \geq 0$

$\Rightarrow 4(1 - 2\sin t)^2 + 4(6\sin t - 5)(1 + 2\sin t) \geq 0$

$\Rightarrow 16\sin^2 t - 8\sin t - 4 \geq 0 \quad \Rightarrow (4\sin^2 t - 2\sin t - 1) \geq 0$

$\Rightarrow 4\left(\sin t - \frac{\sqrt{5}+1}{4}\right)\left(\sin t + \frac{\sqrt{5}-1}{4}\right) \geq 0$

$\Rightarrow \sin t \leq -\left(\frac{\sqrt{5}-1}{4}\right)$ or $\sin t \geq \frac{\sqrt{5}+1}{4}$

$\Rightarrow \sin t \leq \sin(-\pi/10)$ or $\sin t \geq \sin(3\pi/10)$

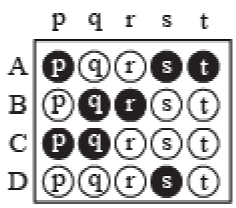
$\Rightarrow t \leq -\pi/10$ or $t \geq 3\pi/10$

(Note that sin x is an increasing function from $-\pi/2$ to $\pi/2$)

$\therefore$ range of t is $[-\pi/2, -\pi/10] \cup [3\pi/10, \pi/2]$.

Match the following of Trigonometric Functions & Equations

DIRECTIONS (Q. 1): Each question contains statements given in two columns, which have to be matched. The statements in Column-I are labelled A, B, C and D, while the statements in Column-II are labelled p, q, r, s and t. Any given statement in Column-I can have correct matching with ONE OR MORE statement(s) in Column-II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example :



If the correct matches are A-p, s and t; B-q and r; C-p and q; and D-s then the correct darkening of bubbles will look like the given.

Q. 1.

In this questions there are entries in columns 1 and 2. Each entry in column 1 is related to exactly one entry in column 2. Write the correct letter from column 2 against the entry number in column 1 in your answer book.

$\frac{\sin 3\alpha}{\cos 2\alpha}$ is (1992 - 2 Marks)

Column I	Column II
(A) positive	(p) $\left(\frac{13\pi}{48}, \frac{14\pi}{48}\right)$
(B) negative	(q) $\left(\frac{14\pi}{48}, \frac{18\pi}{48}\right)$
	(r) $\left(\frac{18\pi}{48}, \frac{23\pi}{48}\right)$
	(s) $\left(0, \frac{\pi}{2}\right)$

Sol.

f $\frac{13\pi}{48} < \alpha < \frac{14\pi}{48} \Rightarrow \frac{13\pi}{16} < 3\alpha < \frac{14\pi}{16}$

and $\frac{13\pi}{24} < 2\alpha < \frac{14\pi}{24}$

$\Rightarrow$ $3\alpha \in$ II quad and $2\alpha \in$ II quad $\Rightarrow$ $\sin 3\alpha = +$ ve

$\cos 2\alpha = -$ ve $\therefore \frac{\sin 3\alpha}{\cos 2\alpha} = -ve$

$\therefore$ (B) corresponds to (p).

If $\alpha \in \left(\frac{14\pi}{48}, \frac{18\pi}{48}\right) \Rightarrow \frac{14\pi}{16} < 3\alpha < \frac{18\pi}{16}$

and $\frac{14\pi}{24} < 2\alpha < \frac{18\pi}{24}$

$\Rightarrow$ $3\alpha \in$ II or III quad and $2\alpha \in$ II quad

$\Rightarrow$ Nothing can be said about the sign of $\frac{\sin 3\alpha}{\cos 2\alpha}$ over this interval.

If $\alpha \in \left(\frac{18\pi}{48}, \frac{23\pi}{48}\right)$ then $\frac{18\pi}{16} < 3\alpha < \frac{23\pi}{16}$

and $\frac{18\pi}{24} < 2\alpha < \frac{23\pi}{24}$

$\Rightarrow$ $3\alpha \in$ III quad and $2\alpha \in$ II quad

$\Rightarrow$ $\sin 3\alpha = -$ ve, $\cos 2\alpha = -$ ve $\therefore \frac{\sin 3\alpha}{\cos 2\alpha} = +ve$

$\therefore$ (A) corresponds to (r)

If a $\in (0, \pi/2)$

$\Rightarrow 0 < 3\alpha < 3\pi/2$ and $2 < 2\alpha < \pi$

$\Rightarrow$ Nothing can be said about the sign of $\frac{\sin 3\alpha}{\cos 2\alpha}$ over the given interval.

Integar Type ques of Trigonometric Functions & Equations

Q. 1. The number of all possible values of q where $0 < \theta < \pi$, for which the system of equations $(y + z) \cos 3\theta = (xyz) \sin 3\theta$

$$x \sin 3\theta = \frac{2\cos 3\theta}{y} + \frac{2\sin 3\theta}{z}$$

$(xyz) \sin 3\theta = (y + 2z) \cos 3\theta + y \sin 3\theta$

have a solution (x_0, y_0, z_0) with $y_0 z_0 \neq 0$, is (2010)

Ans. Sol. (3) The given equations are

$xyz \sin 3\theta = (y + z) \cos 3\theta$ — (1)

$xyz \sin 3\theta = 2z \cos 3\theta + 2y \sin 3\theta$ — (2)

$xyz \sin 3\theta = y + 2z \cos 3\theta + y \sin 3\theta$ — (3)

Operating (1) – (2) and (3) – (1), we get

$(\cos 3\theta - 2 \sin q)y - (\cos 3\theta)z = 0$

and $\sin 3\theta\, y + (\cos 3\theta)z = 0$

which is homogeneous system of linear equation. But

$y \neq 0, z \neq 0$

$$\therefore \frac{\cos 3\theta - 2\sin 3\theta}{\sin 3\theta} = -\frac{\cos 3\theta}{\cos 3\theta} \Rightarrow \cos 3\theta = \sin 3\theta$$

$$\Rightarrow \tan 3\theta = 1 \Rightarrow 3\theta = n\pi + \frac{\pi}{4} \Rightarrow \theta = (4n+1)\frac{\pi}{12}, n \in Z$$

For $\theta \in (0, \pi) \Rightarrow \theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}$

$\therefore$ Three such solutions are possible.

Q. 2. The number of values of q in the interval, $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ **such that** $\theta \neq \frac{n\pi}{5}$ **for n = 0, ±1,±2 and tanθ = cot 5θ as well as sin 2θ = cos 4θ is (2010)**

Sol.

$$\tan\theta = \cot 5\theta, \theta \neq \frac{n\pi}{5}$$

⇒ cos θ cos 5θ – sin5θ sinθ= 0 ⇒ cos 6θ=0

$$\Rightarrow\ 6\theta = \frac{-5\pi}{2},\frac{-3\pi}{2},\frac{-\pi}{2},\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2}$$

$$\Rightarrow\ \theta = \frac{-5\pi}{12},\frac{-\pi}{4},\frac{-\pi}{12},\frac{\pi}{12},\frac{\pi}{4},\frac{5\pi}{12}$$

Again sin 2θ= cos 4θ= 1 – 2 $\sin^2 2\theta$

⇒ $2\sin^2 2\theta$ + sin 2θ –1= 0 ⇒ sin 2θ = $-1\frac{1}{2}$

$$\Rightarrow\ 2\theta = \frac{-\pi}{2},\frac{\pi}{6},\frac{5\pi}{6} \Rightarrow \theta = \frac{-\pi}{4},\frac{\pi}{12},\frac{5\pi}{12}$$

So common solutions are $\theta = \frac{-\pi}{4},\frac{\pi}{12}\text{ and }\frac{5\pi}{12}$

∴ Number of solutions = 3.

Q. 3. The maximum value of the expression

$$\frac{1}{\sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta}$$

Ans. Sol. Let $f(\theta) = \frac{1}{g(\theta)}$

where g (θ) = $\sin^2\theta$+ 3 sinθ cos θ+ 5 $\cos^2\theta$

Clearly f is maximum when g is minimum

Now $(\theta) = \frac{1-\cos 2\theta}{2} + \frac{3}{2}\sin 2\theta + \frac{5}{2}(1+\cos 2\theta)$

$= 3 + 2\cos 2\theta + \frac{3}{2}\sin 2\theta \geq 3 + \left(-\sqrt{4+\frac{9}{4}}\right)$

$\therefore g_{\min} = 3 - \frac{5}{2} = \frac{1}{2} \Rightarrow f_{\max} = 2.$

Q. 4. The positive integer value of n > 3 satisfying the equation

$\frac{1}{\sin\left(\frac{\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)} + \frac{1}{\sin\left(\frac{3\pi}{n}\right)}$ is **(2011)**

Sol. We have, $\frac{1}{\sin\frac{\pi}{n}} - \frac{1}{\sin\frac{3\pi}{n}} = \frac{1}{\sin\frac{2\pi}{n}}$

$\Rightarrow \frac{\sin\frac{3\pi}{n} - \sin\frac{\pi}{n}}{\sin\frac{\pi}{n}\sin\frac{3\pi}{n}} = \frac{1}{\sin\frac{2\pi}{n}} \Rightarrow \frac{2\cos\frac{2\pi}{n}\sin\frac{\pi}{n}}{\sin\frac{\pi}{n}\sin\frac{3\pi}{n}} = \frac{1}{\sin\frac{2\pi}{n}}$

$\Rightarrow 2\sin\frac{2\pi}{n}\cos\frac{2\pi}{n} = \sin\frac{3\pi}{n} \Rightarrow \sin\frac{4\pi}{n} - \sin\frac{3\pi}{n} = 0$

$\Rightarrow 2\cos\frac{7\pi}{2n}\sin\frac{\pi}{2n} = 0 \therefore \Rightarrow \cos\frac{7\pi}{2n} = 0$ or $\sin\frac{\pi}{2n} = 0$

$\Rightarrow \frac{7\pi}{2n} = (2k+1)\frac{\pi}{2}$ or $\frac{\pi}{2n} = 2k\pi$ where $k \in Z$

$\Rightarrow n = \frac{7}{2k+1}$ or $n = \frac{1}{4k}$

($n=\frac{1}{4k}$ not possible for any integral value of k)

As n > 3, for k = 0, we get n = 7.

Q. 5. The number of distinct solutions of the equation
$\frac{5}{4}\cos^2 2x+\cos^4 x+\sin^4 x+\cos^6 x+\sin^6 x=2$ **in the interval [0, 2π] is (JEE Adv. 2015)**

Sol. $\frac{5}{4}\cos^2 2x + \cos^4 x + \sin^4 x + \cos^6 x + \sin^6 x = 2$

$\Rightarrow \frac{5}{4}\cos^2 2x + 1 - \frac{1}{2}\sin^2 2x + 1 - \frac{3}{4}\sin^2 2x = 2$

$\Rightarrow \frac{\pi}{2}(\cos^2 2x - \sin^2 2x) = 0 \Rightarrow \cos 4x = 0$

$\Rightarrow 4x = (2n + 1)\frac{\pi}{2}$ or $x = (2n + 1)\frac{\pi}{8}$

For $x \in [0, 2\pi]$, n can take values 0 to 7

∴ 8 solutions.